Single member actuation in large repetitive truss structures

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Abstract

Numerical results are presented for the energy required to actuate a single member in three large two-dimensional lattice structures. As expected, the actuation energy, relative to the energy required to stretch a bar, scales with bar stockiness to the power two for a hexagonal lattice, whose behaviour is bending dominated, but is not greatly affected by stockiness for a triangulated lattice, whose response is stretching dominated. For a kagome lattice, however, the relative actuation energy scales with stockiness to the power one. Simple models show that, for the kagome, the attenuation of the deformation with distance from the actuated bar itself depends on the stockiness, and this results in the unexpected energy scaling; they also show that the energy is approximately equally partitioned between bending and stretching.
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1. Introduction

Recent work (e.g. Wallach and Gibson, 2001; Deshpande et al., 2001b; Brittain et al., 2001) has shown the feasibility of manufacturing structures that on a micro-scale are repetitive trusses. There is now interest in using the repetitive trusses as the basis for *active* structures (Hutchinson et al., 2003; Donev and Torquato, 2003). Individual actuators replace some of the members of the truss: altering the length of these actuators changes the macroscopic shape of the structure. This paper considers the energy required to actuate a single member of three different designs of large repetitive two-dimensional truss, and in particular shows how this energy scales with the stockiness of the members.

The design of a truss with minimal internal resistance to actuation is made considerably simpler if the truss is assumed to be pin-jointed. Now the way to optimize the structure is clear: the structure should be statically determinate (and hence not allow the build-up of self-stress), and also kinematically determinate (and hence not have any mechanisms). Even if the structure is not pin-jointed, the same structures are also

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likely to be optimal, as the energy of deformation only involves bending energy, which for slender struts requires much less energy than stretching (Deshpande et al., 2001a).

Unfortunately, however, Guest and Hutchinson (2003) have shown that infinite repetitive structures cannot be simultaneously statically and kinematically determinate. This implies that there cannot be a simple and obvious solution to the optimal design for actuation. However, in two dimensions, the kagome truss has been shown to be a promising solution (Hutchinson et al., 2003), as in this case, the existence of internal mechanisms does not have a detrimental impact on the overall stiffness, or the resistance to buckling, of the structure.

This paper will consider the energy penalty associated with actuating large repetitive truss structures by considering the simplest case, the energy required to actuate a single member. The structures examined will not be pin-jointed, as practical manufacturing requires 'welded' joints, rather, the effect of stockiness of the members will be explored.

Three structures will be considered, each based on a different lattice that has all members the same length. The first is a fully triangulated structure; the valence of each node is six, and the structure, when considered as pin-jointed, is clearly statically indeterminate. The second is a hexagonal lattice; the valence of each node is three, and the structure, when considered as pin-jointed, is clearly kinematically indeterminate. Finally the kagome lattice will be considered, which is in some ways an intermediate structure between these other two; the valence of each joint is four.

The paper is structured as follows. In Section 2, the energy associated with actuating simple finite structures will be explored, by way of introduction. Section 3 reports finite element results for the energy penalty associated with actuating a single member in the repetitive truss structures described in the previous paragraph. Section 4 explores simple models that help to explain the finite element results, and Section 5 then concludes the paper.

2. Actuation of simple finite structures

By way of introduction, this section considers actuating single members in the three structures shown in Fig. 1. The structures are chosen so that, if considered to be pin-jointed, one is kinematically indeterminate, one is both statically and kinematically determinate, and one is statically indeterminate. These structures are simple enough to be analysed analytically using standard techniques of structural mechanics; here a simple stiffness matrix formulation was used (Livesley, 1964).

For each structure, all bars are assumed to have a cross-sectional area A, a Young's Modulus E, and a radius of gyration for in-plane bending of k, giving an axial stiffness AE and a bending stiffness AEk^2. These properties are also taken for the member that is actuated, although in practice, actuating members may well
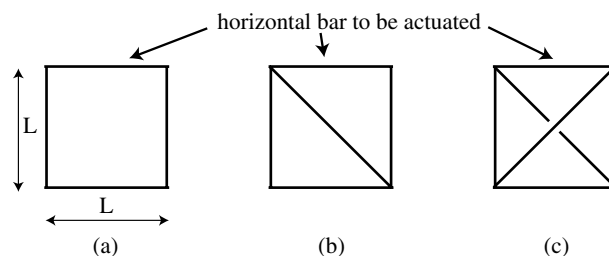

Fig. 1. Three simple structures: for each, the energy required to actuate the top member is calculated. If considered to be pin-jointed, (a) is kinematically indeterminate (it has an obvious shear mechanism), (b) is both statically and kinematically determinate, and (c) is statically indeterminate (it has a state of self stress).

have different elastic properties. The structure is assumed to be not subject to any external loads, and initially unstressed. All calculations are done in a small deformation linear-elastic range. The calculations are carried out for different values of *stockiness*, where the stockiness *s* for each structure is defined as being $s = k/L$, where the length L is shown in Fig. 1. No account is taken of the finite size of the nodes. Note that the definition of stockiness used here is reciprocal to the *slenderness* commonly used in structural mechanics.

The calculation of the work done in actuation proceeds by extending the rest length of the actuated bar by a strain ϵ_a. As the rest of the structure is not completely flexible, this causes a (negative) axial tension in the bar t. The work done by the member during this extension is then $W = -tL\epsilon_a/2$. If the rest of the structure was completely rigid, $t_0 = -AE\epsilon_a$; we take the work done in this case to be a reference energy, $W_0 = (AEL/2)\epsilon_a^2$. In reporting the results of the calculations, we report the non-dimensional $\widehat{W} = W/W_0$. In the context of designing an easily actuated structure, we want $\widehat{W}$ to be small.

The results for $\widehat{W}$ for the three finite structures are shown in Fig. 2. For the two structures (a) and (b) that do not have a pin-jointed state of self-stress, the structure deforms primarily in bending. For a fixed pattern of deformation, the non-dimensionalised bending energy scales with s^2, and for these structures, $\widehat{W}$ scales with s^2.

For structure (c) the results are very different. Here, actuation activates the pin-jointed state of self-stress, and axial deformation is induced. For a fixed pattern of deformation, non-dimensionalised stretching energy does not vary with s, and for this structure, $\widehat{W}$ is approximately constant.

These results reinforce the point made in Section 1, that an actuatable structure should be both statically and kinematically determinate when considered as pin-jointed. For slender structures, both (a) and (b) are easily actuatable. However (a) has a soft shear mode (a mechanism in the pin-jointed case), and this has two implications: first, the structure cannot efficiently carry some loads; and second, the soft mode cannot be actuated by changing the length of any member, as the mode does not involve any of the members changing in length. Hence, for these structures, (b) is optimal.
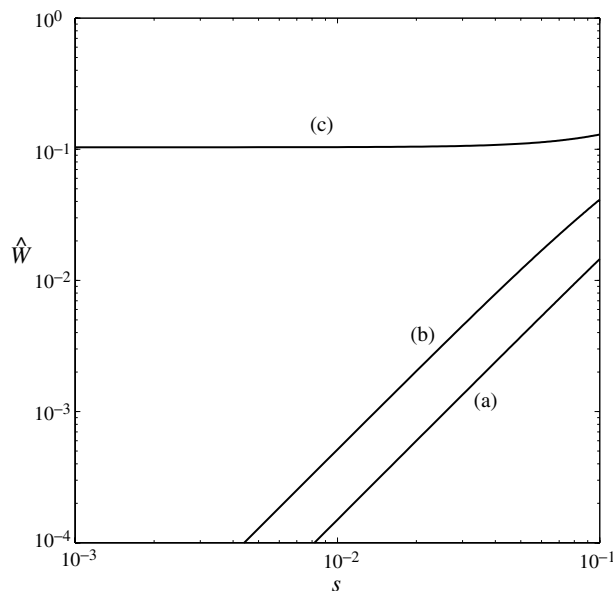

Fig. 2. The energy required to actuate the three structures shown in Fig. 1.

3. Finite-element results for large lattices

This section will explore how the predictable results from Section 2 are affected if the structure that is being actuated is essentially infinite. Finite element models are used to model the actuation of large two-dimensional trusses of three different lattice types: a fully triangulated lattice, a hexagonal lattice, and a kagome lattice. These lattices are shown in Fig. 3. In all cases, the calculations are carried out twice, once with only one node fixed (to suppress rigid-body modes), and once with all boundary nodes fixed. This establishes bounds on the case where the truss is infinite, or, at least, larger than the case considered.

In all cases, the lattice considered is rectangular, with a width of 100 times the individual member length L, and a height of $60 \times L\sqrt{3}/2 \approx 52L$. The lattice member that is actuated is at the centre of the structure, and is parallel with the longer dimension of the structure. The finite element program Abaqus (2002) is used: each bar is modelled with a single Euler–Bernoulli beam element of length L and Young's Modulus E. The cross-section is assumed to be round, of radius r, giving a radius of gyration $k = r/2$; practical structures may well not have round bars, but as only in-plane linear-elastic bending is considered, the actual
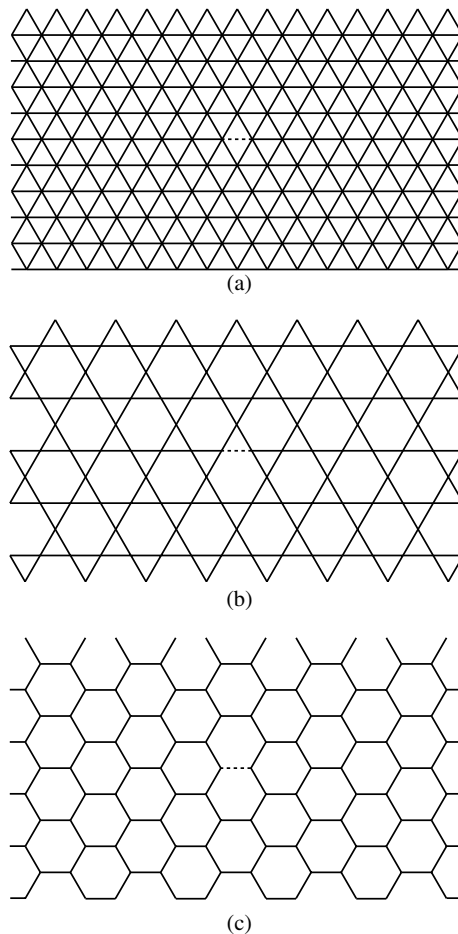

Fig. 3. The central portion of each of the three lattices explored in Section 3; in each case the horizontal strut shown dashed is actuated. (a) A triangulated lattice, (b) a kagome lattice, and (c) a hexagonal lattice.

shape of the bars has no effect on the results. The same properties are also applied to the member that is actuated. The structure is assumed to be not subject to any external loads, and initially unstressed. All calculations are done in a small deformation linear-elastic range.

For each lattice, results for $\widehat{W}$ are calculated for five different bar radii, giving results for $s = k/L = 0.005, 0.015, 0.025, 0.035, 0.045$. These stockiness ratios are chosen as they cover the range of interest for practical applications. For the stockiest bars, the area taken up by nodes will in practice become an important consideration (this is clearly shown later in Fig. 14), but it is not taken into consideration here—all bars are assumed to have length L.

The results of the calculations are plotted in Fig. 4, and the mode of deformation of the structures is shown in Figs. 5–7.

For the triangulated and hexagonal grids, the results are as expected from Section 2. The triangulated grid is clearly statically indeterminate if pin-jointed; its actuation energy is dominated by stretching, and $\widehat{W}$
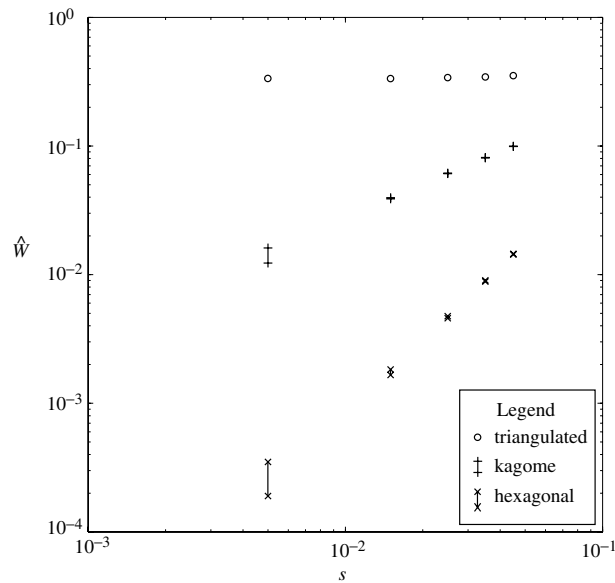

Fig. 4. The energy penalty $\widehat{W}$ associated with actuating a single rod in a large lattice structure. For each stockiness s, and each structure, two results are shown, the upper being with the boundary of the grid fixed, the lower with it free.
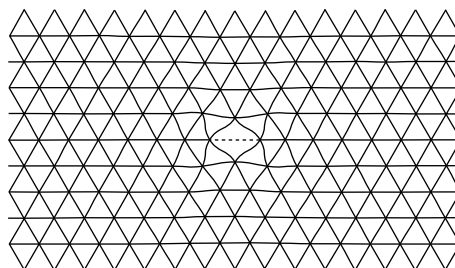

Fig. 5. The shape of the central portion of the triangulated truss after a single rod (shown dashed) is actuated. The calculation assumes small deformations; the deformations shown here are greatly magnified for clarity. The shape shown is for a stockiness ratio $s = 0.005$, but the results for the other stockiness ratios considered are indistinguishable. The width of the bars is not to scale.
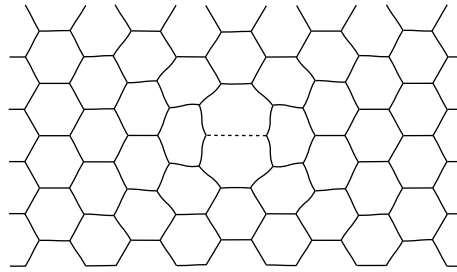
Fig. 6. The shape of the central portion of the hexagonal truss after a single rod (shown dashed) is actuated. The calculation assumes small deformations; the deformations shown here are greatly magnified for clarity. The shape shown is for a stockiness ratio $s = 0.005$, but the results for the other stockiness ratios considered are indistinguishable. The width of the bars is not to scale.

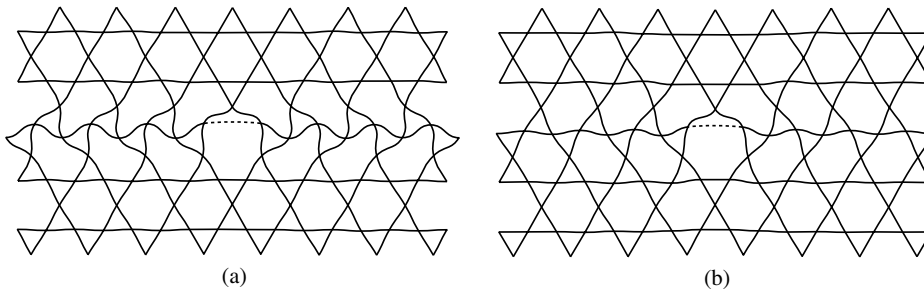(a)   (b)

Fig. 7. The shape of the central kagome truss after a single rod (shown dashed) is actuated. The calculation assumes small deformations; the deformations shown here are greatly magnified for clarity. The width of the bars is not to scale. Stockiness ratio (a) $s = 0.005$, (b) $s = 0.045$. Note that the displacement attenuates more quickly in (b).
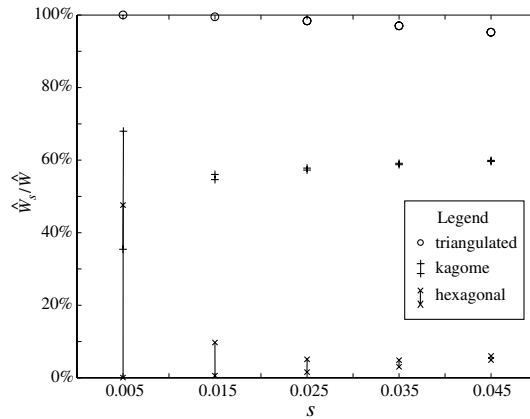

Fig. 8. The proportion of the actuation energy taken up in stretching of bars, $\widehat{W}_s / \widehat{W}$. For each stockiness s, and each structure, two results are shown, the upper being with the boundary of the grid fixed, the lower with it free. Clearly for $s = 0.005$, the grid used is not large enough to give a meaningful result for this quantity.

does not depend greatly on s. The hexagonal grid is clearly kinematically indeterminate, and is not capable of sustaining a state of self-stress if pin-jointed; its actuation energy is dominated by bending, and $\widehat{W}$ scales with s^2.

The results for the kagome truss are unexpected. Here, $\widehat{W}$ scales with s^1. The key to understanding this result is to realise that the assumption made in Section 2 that $\widehat{W}$ is constant for stretching dominated modes, but scales with s^2 for bending dominated modes, depends on the mode of deformation being fixed. For the kagome, however, the actuator is activating what is an infinitesimal mechanism for the pin-jointed case, and it can be seen in Fig. 7 that the distance over which the deformation dies away depends on the stockiness. Thus the mode of deformation itself depends on the stockiness, and simple bending/stretching scaling cannot be applied.

Some insight into the behaviour of the kagome lattice is given by calculating the proportion of the actuation energy that is taken up in stretching, as shown in Fig. 8. As expected, for the triangulated lattice the energy is dominated by stretching, while the hexagonal lattice it is dominated by bending. For the kagome lattice, however, the energy is approximately evenly distributed between bending and stretching.

The next section will present two simple models to further explain the nature of the deformation of the kagome truss, and the scaling of $\widehat{W}$ with s^1.

4. Simple models for the kagome truss

This section will present two simple models, aimed at understanding why the actuation energy for the kagome truss scales with stockiness to the power one.

4.1. A transfer matrix model

The deformed lattices in Fig. 7 shows that the deformation of the kagome structure due to the actuation of a single bar takes place primarily along a narrow corridor parallel with the member being actuated. A simple model can be generated by considering the deformation of the structure shown in Fig. 9, where the actuation is *constrained* to lie in this narrow corridor. Clearly, this model will be too stiff, as the boundaries that are fully constrained would have some flexibility in the true case. However, we would not expect this to affect the trend in behaviour with stockiness. Certainly for the pin-jointed case, *all* deformation occurs along such a narrow corridor. In other cases, the simplified structure will provide a strict upper bound on the energy of actuation.

Because of the one-dimensional repetitive nature of the simplified structure, it is possible to use a *transfer matrix* approach to find the energy of actuation. The details of this calculation are presented in Appendix A. The results are presented in Fig. 12, where they are compared with the finite element results from Section 3, and also the results from the smeared stiffness model described next.

4.2. A smeared stiffness model

The simplified model shown in Fig. 9 makes it clear that the only way that the horizontal displacement of nodes in a horizontal line can attenuate with distance from the actuated bar is through compression of the
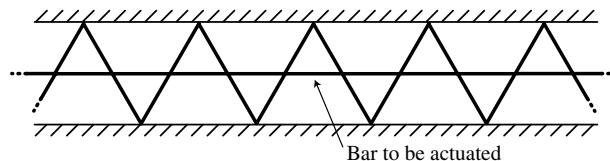

Fig. 9. A simplified model used for calculating the energy of actuation of a kagome truss.
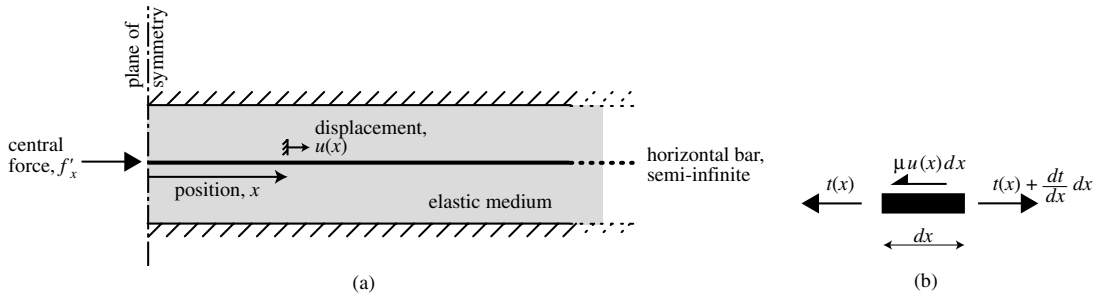
Fig. 10. (a) A 'bar in an elastic material' model. (b) A short element of bar, showing forces applied; μ is the relevant stiffness of the elastic medium, per unit length of the bar. Note that tension is shown positive, even though extension of the actuator causes compressive loads.

horizontal bars; for small deformations, bending cannot change the distance between nodes. Indeed, in the pin-jointed case, the mechanism actuated by the extension of a bar *does not* attenuate.

The observation that horizontal bars must compress suggests that a suitable model would be to consider the response of a bar in an elastic medium, as shown in Fig. 10(a). Here the horizontal bar is preserved, but the rest of the structure is 'smeared' to give a stiffness/unit length of bar of μ. x is taken as the distance from the centre of the actuated bar, where the bar moves horizontally by $u(x)$, and carries a tension $t(x)$. The use of constant μ assumes that the entire structure continues to behave in a linear-elastic manner, as is assumed throughout this paper. The smeared model will be valid as long as the change in $u(x)$ between nodes along the horizontal bar is small—this may be expected when the stockiness of the bars is small, and displacement attenuates slowly, as was seen in Fig. 7(a).

Fig. 10(b) shows a small element of the bar. Equilibrium for this element gives

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \mu u(x) \tag{1}$$

but as the bar remains elastic, with axial stiffness AE, and strain $\mathrm{d}u/\mathrm{d}x$

$$t(x) = AE \frac{\mathrm{d}u}{\mathrm{d}x} \tag{2}$$

Substituting into (1) gives the governing equation for the model

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{\mu}{AE} u(x) \tag{3}$$

for which the general solution, writing $c = \sqrt{\mu/AE}$ is

$$u(x) = P\mathrm{e}^{-cx} + Q\mathrm{e}^{cx} \tag{4}$$

The solution of (4) can be completed by incorporating the boundary conditions. As $x \to \infty$, u remains finite, which implies $Q = 0$. At $x = 0$, we want the displacement be half of the extension of the central bar, $u'_x = \epsilon_a L/2$, and so $P = \epsilon_a L/2$, giving a solution

$$u(x) = \frac{\epsilon_a L}{2} \mathrm{e}^{-cx} \tag{5}$$

To find the work done by the central force f'_x, we have to find an expression for the tension in the rod. Substituting (5) into (2) gives

$$t(x) = \frac{-cAE\epsilon_a L}{2} \mathrm{e}^{-cx} \tag{6}$$

and so

$$f'_x = -t(0) = \frac{cAE\epsilon_a L}{2} \tag{7}$$

Once the central force is known, it is possible to calculate the work done by (both sides) of the central bar as

$$W = \frac{\epsilon_a L f'_x}{2} = \frac{cAE\epsilon_a^2 L^2}{4} \tag{8}$$

and hence the non-dimensional energy $\widehat{W}$ is given by

$$\widehat{W} = \frac{W}{W_0} = \frac{W}{(AEL/2)\epsilon_a^2} = \frac{cL}{2} \tag{9}$$

Thus we have a complete solution for the model, except that we do not have a suitable value for μ and hence c. However, a suitable value can be found by considering the forces required for a uniform displacement of the bar horizontally by u_c. For the smeared model, the force on a bar of length L is given by

$$f = \mu L u_c \tag{10}$$

The equivalent force for the simplified model (Fig. 9), can be found by considering the repetitive deformation shown in Fig. 11. A straightforward structural calculation, now assuming that the bars are axially rigid, gives

$$f = \frac{48EI}{L^3} u_c \tag{11}$$

Equating (10) and (11) gives a suitable value for μ

$$\mu = \frac{48EI}{L^4} = \frac{48EAk^2}{L^4} = \frac{48AEs^2}{L^2} \tag{12}$$

and hence

$$c = \sqrt{\frac{\mu}{AE}} = \frac{4\sqrt{3}s}{L} \tag{13}$$

And so the non-dimensional energy of actuation is given by

$$\widehat{W} = 2\sqrt{3}s \tag{14}$$
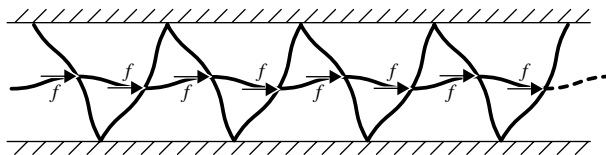
This result is plotted in Fig. 12.

Fig. 11. The simplified kagome model, with all of the central nodes displaced to the right by u_c (and up or down as appropriate by $u_c/\sqrt{3}$, preserving axial lengths).
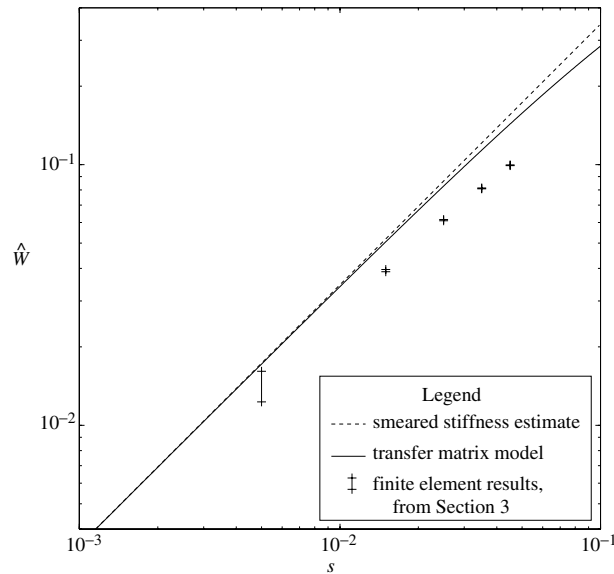
Fig. 12. A comparison of the predictions for the energy required to actuate a single member of a kagome truss.

4.3. Discussion

The results for both of the models described are compared with the computational results in Fig. 12. It can be seen that both models show that the non-dimensionalised energy of actuation scales with s^1. The transfer matrix model actually gives an upper bound on the energy of actuation, as it gives a solution for the case where the nodes around the 'corridor' are fully fixed, rather than having some flexibility. For small stockiness, both models agree.

Both models show that the basic characteristic that leads to $\widehat{W}$ scaling with s^1, which is that the mode of deformation is not fixed, but rather the attenuation of displacement with distance depends on s.

An interesting calculation that can be made with the smeared stiffness model is to work out where the energy of actuation is stored. The energy stored in stretching of the central rod is given by

$$W_{cs} = 2 \times \int_0^\infty \frac{AE}{2} \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x = \int_0^\infty AE \left(\frac{-c\epsilon_a L}{2} \mathrm{e}^{-cx} \right)^2 \mathrm{d}x = \frac{cAE\epsilon_a^2 L^2}{8} = \frac{W}{2} \tag{15}$$

where W is the actuation energy given by (8). Thus the model shows that half of the energy goes into stretching of the *central* rod, and the other half goes into bending of the structure, which agrees with the approximately equal division of bending and stretching energy found from the finite element models, plotted in Fig. 8. In fact, a small proportion of the deformation of the rest of the structure will also be by stretching rather than bending, which explains why the finite element results show that in total, slightly more than half the energy is stored in stretching.

It would not be difficult to make the models presented more accurate. A more complex transfer matrix model could be devised that would take a wider corridor, and hence better model the stiffness. Similarly, a more accurate value of f could be found for the smeared model. But this added complexity would not alter the basic result, that $\widehat{W}$ scales with s^1. Rather, it would lower the energy of actuation predictions to give closer agreement with the numerical results.

5. Conclusion

It is known that, for bending dominated structures, the energy required to actuate a single bar, relative to the energy required to stretch that bar, $\widehat{W}$, scales with stockiness, s, to the power 2, while for stretching dominated structures, $\widehat{W}$ is independent of s. Computations presented here, however, show the interesting result that the energy required to actuate a large kagome truss scales with s^1.

The energy scaling for the kagome truss is anomalous because the mode of deformation itself depends on the stockiness, and simple models can show that this leads to $\widehat{W}$ scaling with s^1. The energy of actuation is approximately equally partitioned between bending and stretching.

The scaling of $\widehat{W}$ with s^1 depends crucially on the existence of a line of straight bars which must compress when one member is extended, and hence the deformation cannot be entirely dominated by bending. However, as the structure deforms, this line of straight bars bends, and it seems likely that the structure will become bending dominated, with a corresponding drop in stiffness. This important non-linear behaviour is the subject of continuing work.

Appendix A. Kagome transfer matrix model

Because of the one-dimensional repetitive nature of the simplified structure shown in Fig. 9, it is possible to use a *transfer matrix* approach to find the energy of actuation (Karpov et al., 2002). Consider a typical section of the structure, shown in Fig. 13. A stiffness matrix can be written that relates displacements at the nodes $\mathbf{d}(n) = [d(n)_x, d(n)_y, \theta(n)]^T$ to the work-conjugate forces $\mathbf{f}(n) = [f(n)_x, f(n)_y, m(n)]^T$, where displacements are only allowed at nodes $n-1$, n, and $n+1$:

$$\begin{bmatrix} \mathbf{K}_0 & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{K}_1^T & \mathbf{K}_0 & \mathbf{K}_1 \\ \mathbf{0} & \mathbf{K}_1^T & \mathbf{K}_0 \end{bmatrix} \begin{bmatrix} \mathbf{d}(n-1) \\ \mathbf{d}(n) \\ \mathbf{d}(n+1) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(n-1) \\ \mathbf{f}(n) \\ \mathbf{f}(n+1) \end{bmatrix} \tag{A.1}$$

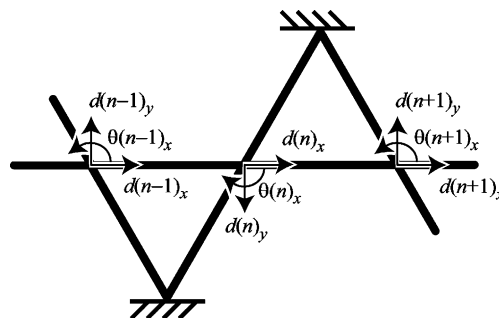The individual matrices $\mathbf{K}_0$ and $\mathbf{K}_1$ can be found by standard methods (Livesley, 1964).

Fig. 13. An element of the kagome simplified model, used for calculating a transfer matrix for the structure. Note the alternating direction of the *y*-direction and rotation at the nodes; this is to take advantage of the plane of *glide reflection* for the structure.

We are interested in the case where the nodal loads are zero; taking the central three equations and setting $\mathbf{f}(n) = \mathbf{0}$ gives

$$\mathbf{K}_1^T\mathbf{d}(n-1) + \mathbf{K}_0\mathbf{d}(n) + \mathbf{K}_1\mathbf{d}(n+1) = \mathbf{0} \tag{A.2}$$

In this case, $\mathbf{K}_1$ is invertible, except in the zero-stockiness limit (when the more advanced methods described in Karpov et al. (2002) can be used), and thus it is straightforward to find the displacements at node $n+1$ in terms of the displacements at nodes n and $n-1$.

$$\mathbf{d}(n+1) = -\mathbf{K}_1^{-1}\mathbf{K}_1^T\mathbf{d}(n-1) - \mathbf{K}_1^{-1}\mathbf{K}_0\mathbf{d}(n) \tag{A.3}$$

Thus we write a state vector

$$\mathbf{x}(n) = \begin{bmatrix} \mathbf{d}(n) \\ \mathbf{d}(n+1) \end{bmatrix} \tag{A.4}$$

and can write a governing equation $\mathbf{x}(n) = \mathbf{H}\mathbf{x}(n-1)$:

$$\begin{bmatrix} \mathbf{d}(n) \\ \mathbf{d}(n+1) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_1^{-1}\mathbf{K}_1^T & -\mathbf{K}_1^{-1}\mathbf{K}_0 \end{bmatrix} \begin{bmatrix} \mathbf{d}(n-1) \\ \mathbf{d}(n) \end{bmatrix} \tag{A.5}$$

where $\mathbf{I}$ is a 3×3 identity matrix. Hence

$$\mathbf{H} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_1^{-1}\mathbf{K}_1^T & -\mathbf{K}_1^{-1}\mathbf{K}_0 \end{bmatrix} \tag{A.6}$$

An eigenvector–eigenvalue decomposition of $\mathbf{H}$ gives useful information about the response of the structure. For this structure, $\mathbf{H}$ has a full set of eigenvectors $\mathbf{u}_1 \ldots \mathbf{u}_6$, with corresponding eigenvectors $\lambda_1 \ldots \lambda_6$ arranged in descending order $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 1 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6$. The eigenvectors come in three sets of reciprocal pairs, $\mathbf{u}_1$ and $\mathbf{u}_6$, $\mathbf{u}_2$ and $\mathbf{u}_5$, and $\mathbf{u}_3$ and $\mathbf{u}_4$. The eigenvalues satisfy $\lambda_1\lambda_6 = \lambda_2\lambda_5 = \lambda_3\lambda_4 = 1$: both eigenvectors in each reciprocal pair represent exactly the same mode of deformation, but in one case decaying when moving from right to left, and in the other case decaying when moving from left to right. For example, the modes associated with eigenvectors $\mathbf{u}_3$ and $\mathbf{u}_4$ are illustrated in Fig. 14. These are the modes that correspond to a mechanism in the rigid-body case; the other modes die away much more quickly.

Because $\mathbf{H}$ has a full set of eigenvectors, any state vector, for instance $\mathbf{x}(1)$, can be written in terms of the eigenvectors, written as a matrix $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6,]^T$:

$$\mathbf{x}(1) = \mathbf{U}\mathbf{c} \tag{A.7}$$

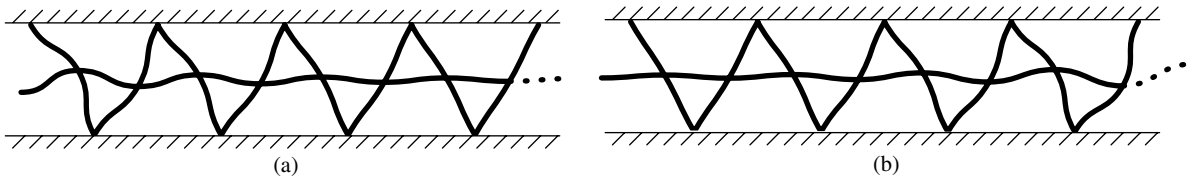where $\mathbf{c}$ is a vector of six unknown coefficients.

Fig. 14. Two of the eigenmodes, (a) $\mathbf{u}_4$ and (b) $\mathbf{u}_3$, for the simplified kagome structure when the stockiness $s = 0.045$. The width of the bars is shown to scale for the case when the bars have a circular cross-section; the bars are in fact rather stocky, and hence the deformation dies away relatively quickly. The finite size of the nodes is neglected, which is obviously rather unrealistic in this case.

Now that the eigenmodes of the structure have been calculated, the next step is to consider appropriate boundary conditions. We shall consider the structure to the right of the actuated bar. As we are considering that the structure is infinite, we want to ensure that we only consider modes that decay, i.e.

$$\mathbf{x}(n) \to 0 \quad \text{as } n \to \infty \tag{A.8}$$

Thus, considering the general form in (A.7), only the coefficients corresponding to the decaying eigenmodes, $\mathbf{u}_4$, $\mathbf{u}_5$ and $\mathbf{u}_6$ will be non-zero. It will be helpful later to partition $\mathbf{U}$ into inflating (i) and decaying (d) modes, and hence we rewrite (A.7) as

$$\mathbf{x}(1) = \begin{bmatrix} \mathbf{d}(1) \\ \mathbf{d}(2) \end{bmatrix} = \begin{bmatrix} \mathbf{U}_{i1} & \mathbf{U}_{d1} \\ \mathbf{U}_{i2} & \mathbf{U}_{d2} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i \\ \mathbf{c}_d \end{bmatrix} \tag{A.9}$$

where the submatrices $\mathbf{U}_{i1}$ etc. are all 3×3.

The next step is to consider appropriate central boundary conditions to calculate the energy stored by actuating one horizontal member. A model for the centre of the structure is shown in Fig. 15. A stiffness matrix can be written that relates displacements at the central node, $\mathbf{d}' = [d_x', d_y', \theta']^{\mathrm{T}}$, and at nodes 1 and 2, to the work-conjugate forces, where displacements are only allowed at these nodes:

$$\begin{bmatrix} \mathbf{K}_2' & \mathbf{K}_1' & \mathbf{0} \\ \mathbf{K}_1'\mathrm{T} & \mathbf{K}_0' & \mathbf{K}_1 \\ \mathbf{0} & \mathbf{K}_1^{\mathrm{T}} & \mathbf{K}_0 \end{bmatrix} \begin{bmatrix} \mathbf{d}' \\ \mathbf{d}(1) \\ \mathbf{d}(2) \end{bmatrix} = \begin{bmatrix} \mathbf{f}' \\ \mathbf{f}(1) \\ \mathbf{f}(2) \end{bmatrix} \tag{A.10}$$

The matrices $\mathbf{K}_0'$, $\mathbf{K}_1'$ and $\mathbf{K}_2'$ can be found by standard methods, and $\mathbf{K}_0$ and $\mathbf{K}_1$ are the same matrices as those given for the general model in (A.1).

We assume that all nodes, except the central node, carry no load. At the central node, we impose a horizontal displacement of half of the extension of the central bar, $d_x' = \epsilon_a L/2$, and symmetry conditions imply that there is no rotation, $\theta' = 0$, and no vertical force, $f_y' = 0$. The displacements at nodes 1 and 2 are given by (A.9), where we only consider the decaying modes, and hence $\mathbf{c}_i = \mathbf{0}$. Thus, (A.10) can be rewritten (without the equations of equilibrium at node 2, which are not required) as

$$\begin{bmatrix} \mathbf{K}_2' & \mathbf{K}_1' & \mathbf{0} \\ \mathbf{K}_1'\mathrm{T} & \mathbf{K}_0' & \mathbf{K}_1 \end{bmatrix} \begin{bmatrix} \begin{pmatrix} \epsilon_a L/2 \\ d_y' \\ 0 \end{pmatrix} \\ \mathbf{U}_{d1}\mathbf{c}_d \\ \mathbf{U}_{d2}\mathbf{c}_d \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} f_x' \\ 0 \\ m' \end{pmatrix} \\ \mathbf{0} \end{bmatrix} \tag{A.11}$$
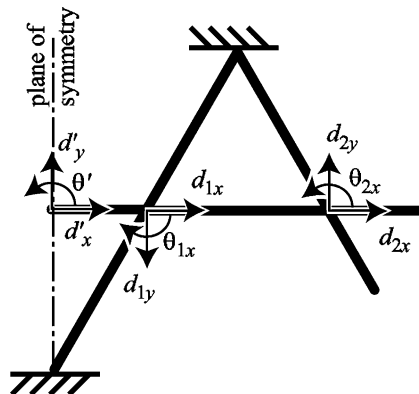

Fig. 15. The centre of the simplified kagome model. Extension of the rest length of the central bar by $\epsilon_a L$ is modelled by setting $d_x' = \epsilon_a L/2$, whilst also imposing suitable symmetry conditions—no rotation, and no vertical force at this central point.

which can be rearranged as

$$\begin{bmatrix} \mathbf{K}_2' & \mathbf{K}_1'\mathbf{U}_{d1} \\ \mathbf{K}_1'\mathbf{T} & \mathbf{K}_0'\mathbf{U}_{d1} + \mathbf{K}_1\mathbf{U}_{d2} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} \epsilon_\mathrm{a}L/2 \\ d_y' \\ 0 \end{pmatrix} \\ \mathbf{c}_d \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} f_x' \\ 0 \\ m' \end{pmatrix} \\ \mathbf{0} \end{bmatrix} \tag{A.12}$$

Eq. (A.12) gives six equations for the six unknowns in the problem, the central horizontal force, f_x', the central moment m', the central vertical displacement, d_y', and the coefficients of the three decaying modes, $\mathbf{c}_d = [c_4, c_5, c_6]^\mathrm{T}$, and can, with some rearrangement, be solved for these unknowns. In particular, it is possible to find the central force, f_x'.

Once the central force is known, it is possible to calculate the work done by (both sides) of the central bar as $E = 2 \times \epsilon_\mathrm{a}Lf_x'/2$. The results of this calculation are presented in Fig. 12.

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